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LAMINAR THEORY OF PERPENDICULAR SHOCKS

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Introduction

This review is confined to the theory of laminar shocks in where there is no reference to instabilities or turbulence. This restriction, plus the limitation to perpendicular shocks, limits the field to a small part of plasma shocks. Nevertheless, this restricted subject is sufficiently ramified to offer a very wide choice of physical mechanisms and a theory that is far from complete.

The original collisionless shock theories[1/] were laminar for the good reason that they antedated all experiments; it is a very appropriate use of Occam's razor to dismiss turbulence until it is forced upon one! Although instability and turbulence are observed in many experiments, they are not necessary for irreversible collisionless heating, and it is very likely that nonturbulent mechanisms will ultimately play a role in the subject.

The key point is that there is an enormous selection of possible irreversible mechanisms within each of the broad categories--collisional, laminar collisionless, and turbulent. Even the classical collisional approach offers so large a variety of mechanisms that the theory is far from complete. An exception is the case of weak shocks where the theory is relatively complete and the complex interplay of collisional mechanisms can be viewed explicitly. Turbulent theories of shock structure can be expected to be as diversified as plasma stability theories. It is certain that many shocks will exhibit a combination of two or more mechanisms.

The significance of shock waves for heating plasma is that the plasma itself decides which is the most effective mechanism. Except for subtleties of bifurcation and non-unique profiles (of which there is, so far, no evidence) the dominant mechanism will be the one which gives the widest profile. Roughly speaking, a theory which predicts too wide a profile is in error, and one which predicts too narrow a profile is incomplete.

We give a brief survey of the subject in terms of the relevant dimensionless parameters; then an account of the highly developed weak shock collisional structure, and of the scattered results which are known in collisionless theory. With regard to dimensionless parameters, the main result is that there are very many. With regard to the collisional theory, a significant point is that experiments are only marginally collisionless, by a factor of about ten; the theoretically estimated laminar thickness may be in error by a comparable amount. The contribution of laminar collisionless theory is, at present, confined to substructures such as the "foot" ahead of a sufficiently strong shock.

Parameters

Counting is the most basic of mathematical operations; counting parameters in a physical problem is the first, and sometimes even the most important mathematical theory. Karl Schindler described the problem of the solar wind flowing past a spherical earth as one with nine dimensionless parameters.[2/] A steady, infinite, plane shock wave is a simpler physical problem with no physical boundaries and no externally imposed length or time scales--it therefore yields "only" six dimensionless parameters (in the simplest version of a fully ionized, two component plasma, with unit ionic charge).

From the molecular constants, e and m, the electromagnetic parameters, $\kappa_o \mu_o = 1/c^2$, the thermodynamic state, n and kT, and the field strength H (six dimensional parameters in all), we can construct one natural mass, length, and time, and three further dimensionless combinations. We can take these to be $\beta = 8\pi p/H^2$, $\omega\tau$, and $\gamma = \omega_p/\omega_c$. To account for ions and electrons, we add $\alpha^2 = m_-/m_+$. These four dimensionless parameters describe a state of thermodynamic equilibrium. For the shock we add a strength parameter and an obliqueness parameter to make six.

In a perpendicular shock we count five instead of six. In a collisionless shock we omit $\omega\tau$; in most collisional theories γ does not appear--therefore four parameters is common. But even the "small" number, four, represents a very high dimensional parameter space, and we can expect qualitatively different modes of behavior to occur as we vary these parameters.

Weak Collisional Shocks

The one case in which there is a relatively complete theory of collisional shocks is that of a weak shock. In this limiting case an explicit solution has been obtained for the system of two Boltzmann (or Fokker-Planck) equations and Maxwell's equations.[3] All parameters other than shock strength are arbitrary.

For the Navier-Stokes equations in an ordinary gas the weak shock profile is given as[4][5]

$$\phi \sim \epsilon \tanh (\epsilon x/L) \tag{1}$$

where ϕ is density, temperature, or velocity, (suitably normalized) and ϵ is the shock strength. The scale length, L, is a sum of two mean free paths, one based on viscosity and the other on heat conductivity.

In the plasma problem, the weak shock profile is given by the same formula (1), except that the scale length L is a very complicated algebraic expression.[3] This expression is greatly simplified by taking a singly charged ionic species and letting $\alpha^2 \rightarrow 0$,[6]

$$L/L_m = \frac{(2b\beta)^{1/2}}{4 + b} \left\{ 3p_\pi^+ + \frac{48b}{13\alpha\beta\omega^2\tau^2} + \left(\frac{72}{13} + 2\beta\right) bQ_1^-/\alpha + 2\beta b Q_1^+ \right\} \tag{2}$$

where

$$\begin{aligned} b &= 3/(6+5\beta) \qquad\qquad Q_1^+ = \frac{2\sqrt{2}}{3K_1}\,\omega^2\tau^2 \\ Q_1^- &= \left(\frac{208}{75} + \frac{192}{25\beta} + \frac{3\alpha^2\omega^2\tau^2}{2\beta} + \frac{13}{24}\,\alpha^2\omega^2\tau^2\right)/K_1 \\ p_{\parallel}^+ &= \frac{5\sqrt{2}}{72}\,\frac{4+(25/18)\alpha^2\omega^2\tau^2}{1+(25/18)\alpha^2\omega^2\tau^2} \\ K_1 &= \frac{5408}{625} + \frac{128}{25}\,\omega^2\tau^2 + \alpha^2\omega^4\tau^4 \end{aligned} \tag{3}$$

and

$$L_m = \frac{3}{4\sqrt{2\pi}}\,\frac{k^2T^2}{ne^4\log Z} \tag{4}$$

is a standard mean free path.

Still further simplification is obtained in the limit $\beta \longrightarrow 0$, $\omega\tau \longrightarrow \infty$; in terms of

$$\begin{aligned} x &= \alpha\omega\tau \\ y &= \beta\omega\tau \end{aligned} \tag{5}$$

we have

$$L/L_m = \frac{2}{9}\sqrt{\beta}\left\{\frac{6}{xy} + \frac{5\sqrt{2}}{18}\,\frac{4+25x^2/18}{1+25x^2/18}\right\} \tag{6}$$

The reason for the complex formula (2)-(3) lies in the complexity of the physical mechanism. There is a 2 x 2 matrix of viscosity coefficients coupling ions and electrons; there is a 3 x 3 matrix of heat conductivity, diffusion (i.e. resistivity), and thermal diffusion coefficients; in addition there are distinct Hall coefficients for ion and electron stresses and for ion and electron heat flows as well as the usual one for electric current. There are more than a dozen distinct regions in the $(\beta, \omega\tau)$ parameter plane where different dissipative mechanisms dominate.[3/]

Figure 1 displays contours, L/L_m = const., for β ranging from .001 to 10 and $\omega\tau$ from 1 to 1000 ($\alpha^2 = 1/3672$ for Deuterium). The scale length, L, ranges from one-hundredth to one thousand times the mean free path in this region. A "standard" calculation of the shock profile, using simple resistivity alone, gives the scale length

$$L_s = c^2/4\pi\sigma U = \frac{\sqrt{b/\beta}}{\alpha\omega^2\tau^2} L_m \tag{7}$$

Figure 2 displays contours of L/L_s for the same range of β and $\omega\tau$ as in Figure 1. The value of L/L_s (ranging from one to ten thousand) can be interpreted as the value of an effective collision frequency, ν_{eff}/ν_c, required to correct the "standard" collisional thickness L_s. The standard formula (7) is frequently used to reduce experimental data, giving a value of ν_{eff}.

The monotone profile (1) results if ϵ is sufficiently small. But this limit is not uniform in parts of the $(\beta, \omega\tau)$-plane,[7] specifically, if, $x = \alpha\omega\tau$ becomes large and $y = \beta\omega\tau$ becomes small as $\epsilon \longrightarrow 0$. Instead of one parameter L, the profile depends on L and on

$$\lambda = (2\epsilon/3)^{1/2} L_p, \; L_p = c/\omega_{pe} \tag{8}$$

For $\mu = \lambda^2/L^2 < 1/8$ the profile is monotone; but it displays damped oscillations when $\mu > 1/8$ or

$$\frac{1}{x} + \frac{5\sqrt{2}}{108} y < 6(\epsilon/3)^{1/2} . \tag{9}$$

Presumably, in a time dependent, steepening wave the profile is determined by the first (i.e. the larger) of the two length scales L and λ. If dispersion, given by λ, interferes before dissipation, measured by L, takes over, the profile oscillates. The dashed lines in Figure 1 mark the regions where oscillatory profiles make their appearance for various values of ϵ.

Another nonuniform limit occurs in the same part of the parameter plane (viz. $x \rightarrow \infty$ and $y \rightarrow 0$) when $\gamma = \omega_p/\omega_c$ becomes small with ϵ. Another oscillatory profile occurs, but this has not yet been calculated quantitatively.[8]

Collisional shock profiles for finite shock strength have only been calculated for a limited range of other parameters and with greatly simplified dissipative mechanisms.[9]

Since the more sophisticated dissipative mechanisms may give rather large values of ν_{eff}/ν_c, the decision that a given experiment is collisionless should never be made solely on the basis of a "standard" resistive thickness or an equivalently crude estimate of the collisional dissipation.

Collisionless Shocks

The universal mechanism for all irreversible behavior is that small changes in initial state become large at some later time. An ordinary molecular collision is an extremely effective mechanism for losing information about the state of an individual particle. Turbulent electric fields can accomplish a similar purpose in "collisionless" shocks. But neither collisions nor turbulence are necessary for irreversibility. The term phase mixing covers a multitude of such effects. In a linear system, collisionless damping is evidenced by the presence of a continuous spectrum--e.g. as in Landau damping. The case of flow perpendicular to the magnetic field is more subtle. Since the linearized equations exhibit a discrete spectrum, dissipation requires a nonlinear analysis. Also, in the transverse steady shock problem it can be shown that two particles which are initially close will never be separated by more than a finite distance. Nevertheless, two particles which are initially close can suffer entirely different histories if, e.g., one is reflected by a potential and the other is not;

(the reflected particle must eventually penetrate the barrier after a second or further return). This mechanism, enhanced by the presence of large electric fields as a consequence of a small mass ratio expansion, has been shown to give rise to a definite irreversible profile, although not a complete shock--specifically through a transition from a constant state to a periodic state of different entropy.[10/] The "foot" which has been observed to develop in front of many medium strength shocks was also theoretically predicted by this analysis.[10/]

One theoretical avenue for an irreversible transition is through a transfer from one branch to another of a multivalued direction field.[11,12/] There is precise agreement in such a case between theory and numerical experiments.[12/]

"Critical" Mach numbers have been observed experimentally across which a qualitative change in structure is observed. There are also several such theoretical transitions, e.g. by a change in the nature of a singularity, but there is no definite correlation of theoretical and experimental transitions as yet.

Although no collisionless theoretical model has yet given a complete transition from one constant state to another, it is very likely that they will play a role in the multistructured shocks that are experimentally observed.

A Model Equation

An equation which displays many of the qualitative features found in most shock waves is[13]

$$U_t + UU_x = U_{xx} + U_{xxx} \qquad (10)$$

All of the coefficients can be set equal to unity by scaling x and t and U; (but this merely transfers the criteria distinguishing different solutions to the initial data).

The two terms $U_t + UU_x = 0$ exhibit steepening, $U_t = U_{xx}$ damping. The combination $UU_x = U_{xx}$ gives a hyperbolic tangent shock as in (1), and the combination of three terms $U_t + UU_x = U_{xx}$ is the Burgers equation which demonstrates the evolution of a monotone shock.[14]

The pair $U_t = U_{xxx}$ demonstrates simple dispersion, $UU_x = U_{xxx}$ gives periodic and solitary wave solutions, and $U_t + UU_x = U_{xxx}$ is the Korteweg de Vries equation which describes nonlinear but reversible interaction between waves.[15]

The complete steady state, $UU_x = U_{xx} + U_{xxx}$ is the equation mentioned previously which describes a family of steady shock profiles ranging from monotone to damped oscillatory. One question that could be answered by a study of (10) is the interplay between dissipation and dispersion in a steepening wave, as previously mentioned.

The initial conditions are essential. For example, an initial state which takes the same constant value at $x = \pm \infty$ would eventually decay[16/]; an initial state with different constant values at $x = \pm \infty$ should develop into a shock. It would be interesting to study the evolution of solutions of the Korteweg-de Vries equation for an initial state joining two different constant states.

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References

1. C. Gardner, H. Goertzel, H. Grad, C. Morawetz, M. Rose, H. Rubin, Proc. of Second Int'l. Conf. on the Peaceful Uses of Atomic Energy, 31, (Geneva 1958), p.230.
2. K. Schindler, Proc. of a Study Group Held at the European Space Research Institute, ESRO-SP 51 (Frascati 1969), p.35.
3. P. N. Hu, Phys. Fluids 9, 89 (1966).
4. G. I. Taylor, Proc. Roy. Soc., A84 (1910).
5. H. Grad, Comm. Pure and Appl. Math. 5, 257 (1952).
6. See Appendix, Ref. 7 - a numerical error in this formula has been corrected.
7. H. Grad, P. N. Hu, Phys. Fluids 10,2956 (1967).
8. H. Grad, P. N. Hu, to be published
9. For example see W. Marshall, Proc. Roy. Soc. A233, 367 (1955) and P. Germain, Rev. Mod. Phys. 32, 951 (1960).
10. C. S. Morawetz, Phys. Fluids 4, 988 (1961).
11. A Banos Jr. and A. R. Vernon, Nuovo Cimento 15, 269 (1960).
12. K. W. Morton, J. Fluid Mech. 14, 369 (1962).
13. This is mentioned in C. H. Su and C. S. Gardner, J. Math. Phys. 10, 536 (1969).
14. For example, see E. Hopf, Comm. Pure and Appl. Math. 3, 201 (1950).
15. For example, see N. J. Zabusky, Proc. of the Symposium on Nonlinear Partial Differential Equations, W. Ames, Ed. (Academic Press Inc., New York, 1967), p.223.
16. E. Ott and R. N. Sudan, "Damping of Solitary Waves", Cornell University Laboratory of Plasma Studies, Report LPS 30, September 1969.

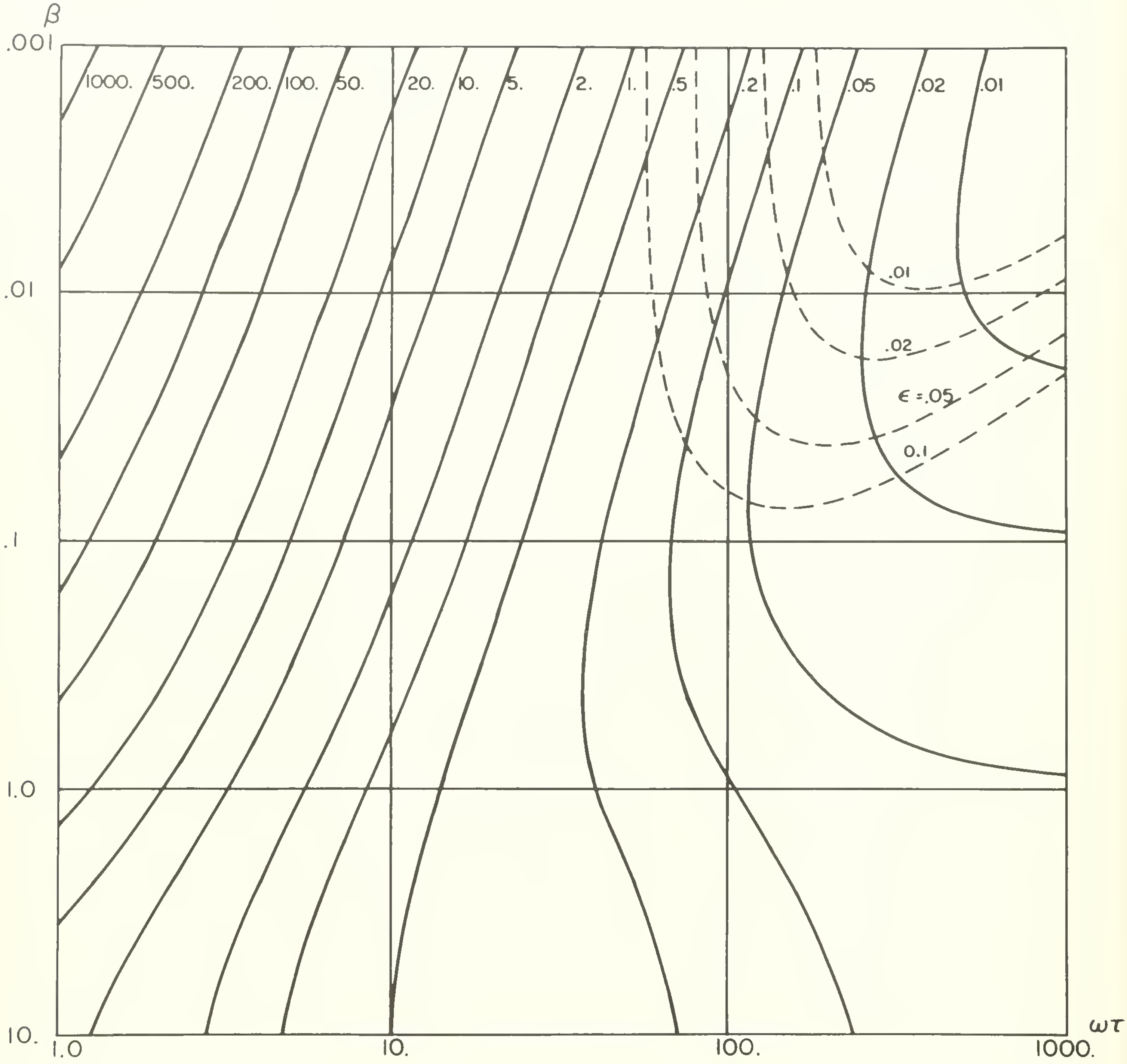

Fig. 1 Shock Thickness as a function of β and ωτ for Deuterium ($\alpha^2 = 1/3672$)

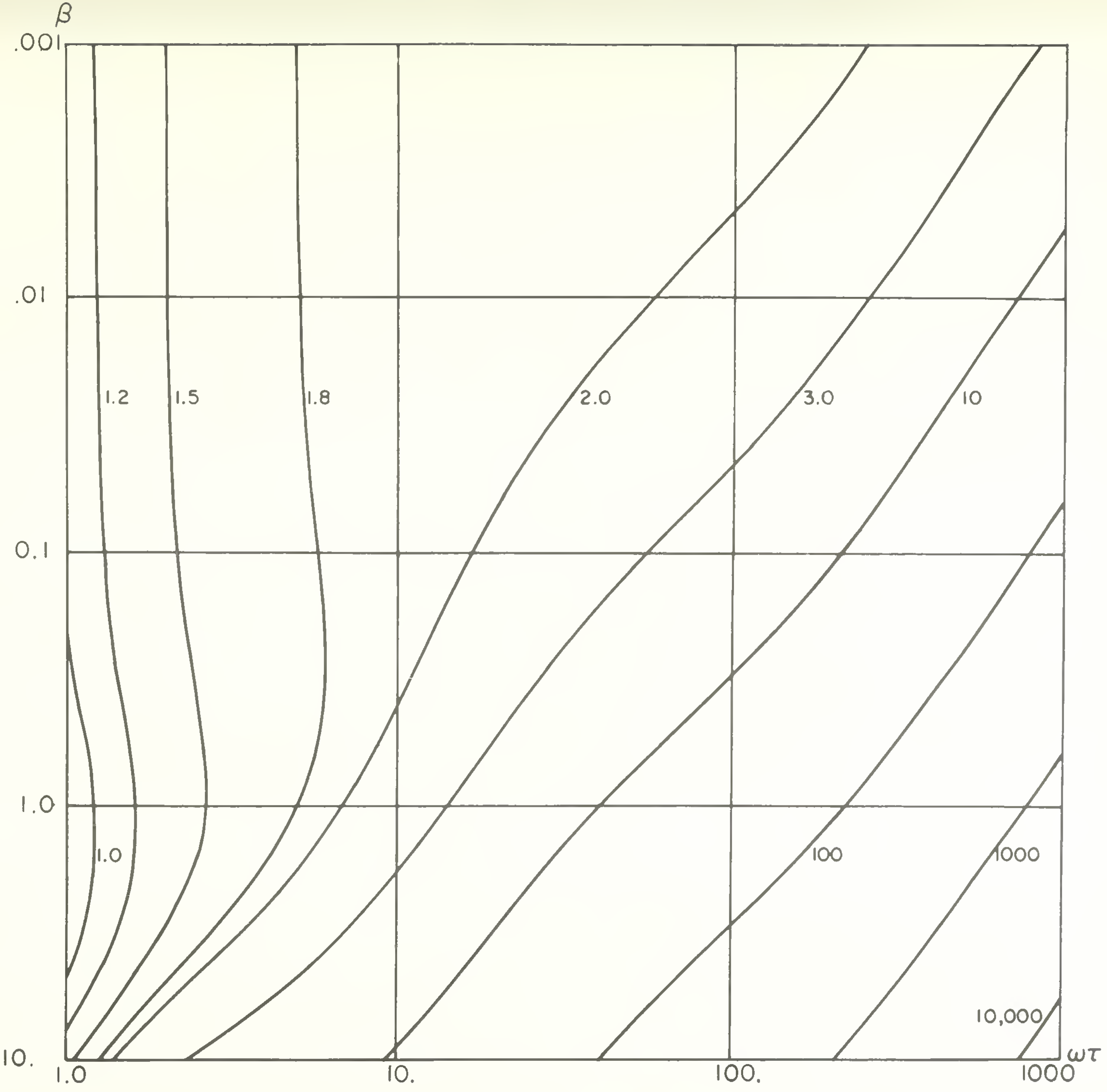

Fig. 2 Effective Collision Frequency

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